Modelling and Information Entropy of Design Spaces

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Extended System Simulation

• Connectivity, co-simulation, multi-core, FMU etc.
• Simulation based optimization
• Design analytics
  – I.e. sensitivity analysis, correlation analysis, robustness, complexity metrics, etc.
  – Methods for experimental validation
• Parametrization for design.
  – Analytic parametrization, and reduction
• Test case modelling and management
Extended System Simulation

Test case model

Parametrization, Design space

Simulation model

Optimization

Objective function

Design Analytics
Information Theory

• Claude Shannons groundbreaking work.

Information at source

Transmitter

Information at receiver

Receiver

Noise

https://youtu.be/R4OlXb9aTvQ
Applications

- Product Platforms
- Complexity of Computer Programs
- Logic Hardware Design
- Human factors
- Search Theory
- Axiomatic Design
- System Design
- Optimization
Amount of Information content (Information Entropy)

The differential information entropy for continuous signals, defined by Shannon [1] as:

\[ H = -\int_{-\infty}^{\infty} p(x) \log_2 (p(x)) dx \]

Kullback-Leibler divergence

\[ H_{rel} = -\int_{-\infty}^{\infty} p(x) \log_2 \left( \frac{p(x)}{m(x)} \right) dx \]

Generalized

\[ H_{rel} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x_1, \ldots, x_n) \log_2 \left( \frac{p(x_1, \ldots, x_n)}{m(x_1, \ldots, x_n)} \right) dx_1 \ldots dx_n \]
Amount of Information content
(Information Entropy)

If the distribution \( m(x) \) is a rectangular distribution in the bounded interval.

\[
I_x = H_{rel}(x) = - \int_{x_{\text{min}}}^{x_{\text{max}}} p(x) \log_2 \left( \frac{p(x)}{x_R} \right) dx
\]

Generalized

\[
I_x = - \int_{x_{1,\text{min}}}^{x_{1,\text{max}}} \int_{x_{n,\text{min}}}^{x_{n,\text{max}}} p(x_1, \ldots, x_n) \log_2 \left( \frac{p(x_1, \ldots, x_n)}{x_{R_1} \cdots x_{R_n}} \right) dx_1 \cdots dx_n
\]

More compact

\[
I_x = \int_D p(x) \log_2 \left( \frac{p(x)}{D} \right) dD
\]
Amount of Information content (Information Entropy)

The information content $I$ of a variable (in bits).

$$I = - \int_{x_{\text{min}}}^{x_{\text{max}}} p(x) \log_2 \left( p(x) x_R \right) dx \quad x_R = x_{\text{max}} - x_{\text{min}}$$

If the range $x_r$ is divided in equal parts $\Delta x$ the amount of information is:

$$I = \log_2 \frac{x_r}{\Delta x} = \log_2 \frac{1}{\delta_x}$$

Here $\Delta x$ is the tolerance in $x$. 

or more general:

$$I = \log_2 \frac{S_0}{S}$$
The choice of logarithm as a base (Shannon)

- It is practically more useful. Parameters of engineering importance such as time, bandwidth, number of relays, etc., tend to vary linearly with the logarithm of the number of possibilities. For example, adding one relay to a group doubles the number of possible states of the relays. It adds 1 to the base 2 logarithm of this number. Doubling the time roughly squares the number of possible messages, or doubles the logarithm, etc.

- It is nearer to our intuitive feeling as to the proper measure. This is closely related to (1) since we intuitively measures entities by linear comparison with common standards. One feels, for example, that two punched cards should have twice the capacity of one for information storage, and two identical channels twice the capacity of one for transmitting information.

- It is mathematically more suitable. Many of the limiting operations are simple in terms of the logarithm but would require clumsy restatement in terms of the number of possibilities.
Design Information Entropy is a Measure of the Size of the Design Space

*Lego example*

- The design space of a set of Lego bricks represents all combinations of arranging these bricks.
- With a set of only two bricks with four knobs on each there are 51 discrete possible arrangements.
- Two of these represents picking only one brick. And one state is to pick no one.
- The 51 different configuration (states) means that the amount of information needed to specify a particular design is:

\[ I_x = \log_2 n_{Dstate} = \log_2 51 = 5.7 \text{bits} \]
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\]
Design Space with Both Continuous and Discrete Variables

- The position of the inserted axis represents a continuous variable.
- The information entropy associated with that is dependent on the accuracy with which it is specified.

\[
I_x = \log_2 n_{Dstates} + \log n_{Cstates} + \log_2 \frac{x_R}{\Delta x}
\]

- The axis can be in three positions and if the position of the axis within one hole is specified within 10% the total information entropy is:

\[
I_x = \log_2 (51+3) + \log_2 \frac{1}{0.1} = 8.2 \text{bits}
\]
Design information entropy can be used as a measure of quality of product platforms for modular design. A good product platform should have little “waste” of design space.
Design Information

- Design information entropy
- The amount of bits needed to specify a design within a design space.
- To specify one design of the four takes:

\[
H_x = \log_2 \frac{S}{s} = \log_2 n_s = \log_2 (2 \times 1 \times 2) = 2 \text{ bits}
\]
Design Information

- To specify one design of the four takes:
  \[ H_x = \log_2 \frac{S}{s} = \log_2 n_s = \log_2 (2 \times 1 \times 2) = 2 \text{ bits} \]

- The entropy of the constraint design space is
  \[ H_c = \log_2 n_v = \log_2 3 = 1.58 \]

- Wasted design information entropy can be formulated as:
  \[ H_w = H_x - H_c = -\log_2 \frac{S_x / s}{S_c / s} = -\log_2 \frac{S_x}{S_c} = 0.42 \]
Morphological Matrix for concept selection

\[ N_s = \prod_{i=1}^{n_f} n_{m,i} \]

\[ N_s = \prod_{i=1}^{4} 3 = 3^4 = 81 \]

\[ H_x = \log_2 N_s = \log_2 81 = 6.34 \text{ bits} \]

Increasing the number of rows by one:

\[ N_s = \prod_{i=1}^{4} 3 = 3^5 = 243 \]

\[ H_x = \log_2 N_s = \log_2 243 = 7.92 \text{ bits} \]

The entropy increases linearly with number of rows.

Information Entropy gives a measure of complexity more consistent with experience!
Information Entropy of Morphological Matrix

In the general case there can be variable number of elements in each row.

\[ N_s = \prod_{i=1}^{4} 4 \times 2 \times 3 \times 2 \times 3 = 144 \]

\[ H_x = \log_2 N_s = \log_2 144 = 7.16 \text{ bits} \]
Information Entropy and Complexity

According to Axiomatic Design the best designs are uncoupled

\[
\begin{pmatrix}
FR_1 \\
FR_2
\end{pmatrix} = \begin{pmatrix}
X & 0 \\
0 & X
\end{pmatrix}\begin{pmatrix}
DP_1 \\
DP_2
\end{pmatrix}
\]

Functional requirements (Anatomy)  Design parameters (Architecture)

If this is true. Design decision becomes independent of each other
Example: UAV Aircraft Concept Generation

![Image of UAV Concepts]

<table>
<thead>
<tr>
<th>Design elements</th>
<th>Alternative solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal stabilization</td>
<td>Front (canard) Aft Aft tail integrated Wing integrated</td>
</tr>
<tr>
<td>Vertical stabilization</td>
<td>Central Wing tip Aft tail integrated Upper Lower</td>
</tr>
<tr>
<td>Tail mount</td>
<td>Single fuselage Twin boom</td>
</tr>
<tr>
<td>Propulsion</td>
<td>Tractor Pusher</td>
</tr>
</tbody>
</table>

\[
N_s = \prod_{i=1}^{n_f} n_{m,i} \quad n_s = 4 \times 5 \times 2 \times 2 = 80
\]

\[
I_x = -\log_2 \left( \frac{1}{N_s} \right) = -\log_2 \left( \frac{1}{80} \right) = 6.32 \text{ bit}
\]
Aircraft Optimization

• For the aircraft example typical design parameters would be;
  – wing span, root cord, tapering, thickness, and sweep, structural weight, fuel weight, engine size, wing position, span of horizontal tail, cruise speed.
Design uncertainty as a function of design information

\[ I_x = \log_2 \frac{x_R}{\Delta x} = \log_2 \delta_x \]

\[ \delta_x = 2^{-I_x} \]
Information entropy of design $s$ relative to design space $S$

$$H_x = \log_2 \frac{S}{s}$$

Information entropy to measure convergence in optimization
Information increase in Optimization

Information entropy is estimated as

$$\hat{H}_x = -n \log_2 \left( \max \left( \delta_{x,i} \right) \right)$$

$$\delta_{x,i} = \frac{x_{i,\text{max}} - x_{i,\text{min}}}{x_{0,i,\text{max}} - x_{0,i,\text{min}}}$$

Figure 5. Accumulation of information as a function of number of objective function evaluations
Meta object function

\[ I_{tot} = (1 - P_{opt}) \log_2 \left( \frac{1 - P_{opt}}{1 - \varepsilon_x^n} \right) + P_{opt} \log_2 \left( \frac{P_{opt}}{\varepsilon_x^n} \right) \]

\[ \phi^{(2)} = \frac{I_x}{N_m} = \frac{1}{N_m} \left( (1 - P_{opt}) \log_2 \left( \frac{1 - P_{opt}}{1 - \varepsilon_x^n} \right) + P_{opt} \log_2 \left( \frac{P_{opt}}{\varepsilon_x^n} \right) \right) \]

expresses the total uncertainty, representing the sum of uncertainty in location and uncertainty of success.
Information Entropy in System Design
A system configuration is defined by its components and by how they are connected. The design space can thus be expressed as:

\[ N_T = N_s \times N_p \]  \hspace{1cm} (9.35)

Here \( N_s \) is the number of possibilities for component selections, and \( N_p \) the number of possible ways to connect the components. The corresponding information entropy is:

\[ I_T = \log_2 N_T = \log_2 N_s + \log_2 N_p \]  \hspace{1cm} (9.36)
Information Entropy in System Design

\[ N_s = 5^4 = 625 \]
\[ I_s = \log_2 N_s = 9.29 \text{bit} \]
Information Entropy in System Design

Assuming apriori information that the system should contain one cylinder one pump and one tank and a library with variants of these

\[ N_s = N_{cyl} \times N_{valve} \times N_{pump} \times N_{tank} = 6 \times 27 \times 6 \times 1 = 972 \]  \hspace{1cm} (9.43)

and hence

\[ I_s = \log_2 N_s = \log_2 972 = 9.92 \]  \hspace{1cm} (9.44)
Connectivity

<table>
<thead>
<tr>
<th>System components</th>
<th>Connectors</th>
<th>A</th>
<th>B</th>
<th>A</th>
<th>B</th>
<th>P</th>
<th>R</th>
<th>P</th>
<th>R</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>Piston</td>
<td>A</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td>Servo valve</td>
<td>A</td>
<td>1</td>
<td></td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td></td>
<td>1</td>
<td>X</td>
<td>X</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
<td>X</td>
<td>X</td>
<td>X</td>
<td></td>
</tr>
<tr>
<td></td>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>X</td>
</tr>
<tr>
<td>Pump</td>
<td>P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Tank</td>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ I_p = \frac{n_p \times n_p}{2} - n_p \]  
\[ (9.45) \]

Here \( n_p \) is the total number of ports in the system. For this example it is:

\[ I_p = \frac{10 \times 10}{2} - 10 = 40 \]  
\[ (9.46) \]

The total amount of information is:

\[ I_x = I_s + I_p \]  
\[ (9.47) \]

\[ I_x = 9.92 + 40 = 49.92 \text{bits} \]  
\[ (9.48) \]

Figure 9.11: Connectivity matrix of a hydraulic servo.
Growth of design information entropy during the design process

\[ I_R = \log_2 \frac{S'_0}{S_0} \]  

Design space generation

\[ I_t = -\log_2 \frac{\sum_{i=1}^{m} S_i}{S_0} \]  

Concept generation

\[ I_{II} = -\log_2 \frac{\sum_{i=1}^{n} S_{j(i)}}{\sum_{i=1}^{m} S_i} \]  

Concept screening

\[ I_{III} = -\log_2 \frac{S_\delta}{\sum_{i=1}^{n} S_{j(i)}} \]  

Concept optimization and selection
Design space expansion

- The design information entropy can be increased in two ways
  - Refinement
  - Design space expansion

- Design space can be increased in several ways like:
  - Adding more bricks
  - Adding other types of bricks
  - Releasing more design parameters in a design

\[
I'_x = \log_2 \frac{x'_R}{\Delta x} = \log_2 \left( \frac{n \times x_R}{\Delta x} \right) = \log_2 n + \log_2 \left( \frac{x_R}{\Delta x} \right) = \log_2 n + I_x
\]

\[
I'_x = \log_2 \frac{x'_R}{\Delta x'} = \log_2 \left( \frac{x_R}{\Delta x} \right)^n = n \log_2 \frac{x_R}{\Delta x} = nI_x
\]
# Design Space Generation and Parameter Reduction

**Example: Electric Motor Data**

<table>
<thead>
<tr>
<th>Voltage [V]</th>
<th>max power [W]</th>
<th>speed at load [rad/s]</th>
<th>max torque [Nm]</th>
<th>volume [cm³]</th>
<th>mass [kg]</th>
<th>power intensity $k_p$ [kW/kg]</th>
<th>mean pressure (torque density) $p_m$ [bar]</th>
<th>power density $\rho_p$ [W/cm³]</th>
<th>torque intensity $k_T$ [Nm/kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>8.4</td>
<td>210</td>
<td>1068</td>
<td>0.20</td>
<td>55</td>
<td>0.1716</td>
<td>1.22</td>
<td>0.04</td>
<td>3.84</td>
<td>1.15</td>
</tr>
<tr>
<td>8</td>
<td>320</td>
<td>1378</td>
<td>0.23</td>
<td>54</td>
<td>0.29</td>
<td>1.10</td>
<td>0.04</td>
<td>5.95</td>
<td>0.80</td>
</tr>
<tr>
<td>24</td>
<td>609</td>
<td>523</td>
<td>1.164</td>
<td>343</td>
<td>1.10</td>
<td>0.55</td>
<td>0.03</td>
<td>1.78</td>
<td>1.06</td>
</tr>
<tr>
<td>24</td>
<td>1440</td>
<td>450</td>
<td>3.2</td>
<td>729</td>
<td>2.4</td>
<td>0.60</td>
<td>0.04</td>
<td>1.98</td>
<td>1.33</td>
</tr>
<tr>
<td>24</td>
<td>3580</td>
<td>471</td>
<td>7.6</td>
<td>729</td>
<td>3.9</td>
<td>0.92</td>
<td>0.10</td>
<td>4.91</td>
<td>1.95</td>
</tr>
<tr>
<td>50</td>
<td>15992</td>
<td>419</td>
<td>38.20</td>
<td>4539</td>
<td>9.36</td>
<td>1.71</td>
<td>0.08</td>
<td>3.52</td>
<td>4.08</td>
</tr>
<tr>
<td>460</td>
<td>73763</td>
<td>175</td>
<td>420.74</td>
<td>23487</td>
<td>215.00</td>
<td>0.34</td>
<td>0.18</td>
<td>3.14</td>
<td>1.96</td>
</tr>
<tr>
<td>460</td>
<td>198499</td>
<td>215</td>
<td>922.87</td>
<td>86524</td>
<td>215.00</td>
<td>0.92</td>
<td>0.11</td>
<td>2.29</td>
<td>4.29</td>
</tr>
<tr>
<td>460</td>
<td>491751</td>
<td>175</td>
<td>2813.33</td>
<td>165518</td>
<td>907.00</td>
<td>0.54</td>
<td>0.17</td>
<td>2.97</td>
<td>3.10</td>
</tr>
</tbody>
</table>

**Average values**
- 0.88
- 0.09
- 3.38
- 2.19
Power to weight relation
(electric motor)

![Graph showing power to mass relation](image)

Power to mass relation

\[ y = 1.0832x + 0.0542 \]

\[ R^2 = 0.9906 \]
Principal Component Analysis
to minimize waste of design space (using Singular Value Decomposition, SVD)
The meaning of the matrices

\[ X = U \times W \times V^T \]

Adding logarithmic scaling and removal of offset (set mean to zero) means that limits in U on set to -1 and 1 covers the data set within one standard deviation, and thus results points likely to be feasible.
# SVD model of Electric motors

<table>
<thead>
<tr>
<th></th>
<th>Reliance K22M13031</th>
<th>Estimate</th>
<th>Adjusted</th>
<th>Result</th>
<th>Average</th>
<th>SVD variables</th>
<th>w-diagonal</th>
<th>residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corner power [W]</td>
<td>491751</td>
<td>491744</td>
<td>5.69</td>
<td>1.99</td>
<td>3.70</td>
<td>-1.163</td>
<td>0.130</td>
<td>-0.001</td>
</tr>
<tr>
<td>Speed at load [rad/s]</td>
<td>175</td>
<td>175</td>
<td>2.24</td>
<td>-0.46</td>
<td>2.70</td>
<td>0.331</td>
<td>0.122</td>
<td>-0.021</td>
</tr>
<tr>
<td>Max Torque [Nm]</td>
<td>2813</td>
<td>2813</td>
<td>3.45</td>
<td>2.44</td>
<td>1.01</td>
<td>-1.494</td>
<td>0.008</td>
<td>-0.001</td>
</tr>
<tr>
<td>diameter [mm]</td>
<td>446</td>
<td>446</td>
<td>2.65</td>
<td>0.63</td>
<td>2.02</td>
<td>-0.384</td>
<td>0.005</td>
<td>0.064</td>
</tr>
<tr>
<td>volume [cm³]</td>
<td>165518</td>
<td>165516</td>
<td>5.22</td>
<td>2.06</td>
<td>3.16</td>
<td>-1.256</td>
<td>-0.018</td>
<td>0.084</td>
</tr>
<tr>
<td>mass [kg]</td>
<td>907</td>
<td>907</td>
<td>2.96</td>
<td>2.21</td>
<td>0.75</td>
<td>-1.298</td>
<td>-0.079</td>
<td>-0.085</td>
</tr>
</tbody>
</table>

**Diagram:**

- **w-diagonal**

![w-diagonal diagram](image)
Example of Design Space for Parametrization: Aircraft Wing Planform

\[ S = \frac{b}{2} (c_r + c_t) \]
\[ AR = \frac{b^2}{S} \]
\[ c_t = c_r \lambda \]
Design space of alternative parameters, with log-axis

\[
\log_2(c_t) \\
\log_2(c_r) \\
\log_2(b)
\]
Design space of SVD-parameters, with log-axis
## Design Space Volumes

<table>
<thead>
<tr>
<th>Parameter set</th>
<th>Design space volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>b, cr, ct</td>
<td>0.914</td>
</tr>
<tr>
<td>S,AR,l</td>
<td>0.877</td>
</tr>
<tr>
<td>SVD</td>
<td>0.25</td>
</tr>
</tbody>
</table>

\[
H_w = H_1 - H_2 = \log_2 \frac{S_1}{S_2} = \log_2 \frac{S_1}{S_2} = \log_2 \frac{0.914}{0.25} = 1.87 \text{bits}
\]
Conclusions

- **Design information entropy** represents a measure of the precision by which a design is defined relative to the **design space** in consideration. It is also proportional to the dimensionality of the design problem.
- Design information entropy can be used as one measure of **complexity**.
- “Thinking outside the box” is the task of finding useful directions to expand the design space.
- **Analytical parametrization** through SVD of a design can be made using sample designs to span the design space. It can also be used to produce scaling models of components. In some sense it can be regarded as the **ideal parameter set**.